Jl. of Computers in Mathematics and Science Teaching (2005) **24**(2), 179-196

Learning the Basics with Calculus

WALTER M. STROUP
The University of Texas at Austin
USA
wstroup@mail.utexas.edu

A critical issue related to the use of simulation environments in school-based learning is the relationship between the use of these environments and the structure of traditional curricula. Teachers are often caught in the bind of attempting to address significantly raised standards for what all students should know, yet receiving very little support to innovate in their use of simulations in their approaches to teaching. Particularly challenging for practitioners and school-based researchers alike is the expectation that students should first master "the basics" before engaging innovation related to learning advanced content such as calculus. This paper summarizes years of work related to the early learning of the mathematics of change. The goal is to resolve this tension by suggesting that advanced mathematics, embodied in an accessible simulation environment, helps situate and deepen students' understanding of the basics as they typically are encountered in the traditional mathematics curriculum. Selected episodes transcribed from work in economically challenged schools in the United States are used to illustrate the practical potential of this reconsideration of the relationships between basic and advanced topics. The delta-blocks approach to the mathematics of change, making use of widely available forms of technology and the *MathWorlds* simulation software, is introduced and illustrated.

INTRODUCTION

Traditionally, we think of calculus as a culminating course in a secondary mathematics curriculum. It seems backward, then, to suggest that calculus can help our younger students make better sense of topics we typically label "the basics." If anything, we assume in our curricula and in our teaching that calculus is a subject to be studied well after the basics are mastered and only after a long series of prerequisite coursework has been taken. As a result, most of our students do not progress as far as calculus; this limits them in terms of their opportunities in post-secondary education. It also limits them in terms of the formal mathematical tools they can bring to situations where rate varies. The world in which we live is dynamic and changing and all our students should develop powerful ways of talking about change.

However, what if our assumption about the place of calculus in the curriculum was not the norm? What if students, using a powerful simulation environment, could master significant aspects of calculus reasoning well before they finish middle school? What if this could be accomplished, not by adding to an already crammed curriculum, but by leveraging time used to engage some of the basics of the late elementary and early middle school curriculum? In addition, what if this all could be accomplished using technologies—either TI-83+ calculators or standard computers—running software that is either free or available at a very minimal per-seat cost?

This article briefly introduces just such a possibility using approaches and technologies that were developed in richly diverse, and yet economically challenged schools in Massachusetts, Texas, New Jersey, and Utah with significant support from the National Science Foundation and Bob Moses' Algebra Project. We provide an overview of what is called the "delta-blocks" approach to learning the math of change. In so doing we also provide some examples of how this approach is useful in helping our students make sense of some of the basics in our curricula. These basics include signed addition and subtraction, the area model of multiplication and division, geometric area, graphing, and graph interpretation.

Delta Blocks and Embodying the Area Under a Rate Graph

The *delta-blocks* approach to learning the math of change (Stroup, 1996; Stroup, 2002) was originally developed in late elementary, middle, and high school classrooms as part of work on the SimCalc project. This project had a commitment to investigate starting from the integral and then

moving to the derivative as part of what distinguished it from previous early-calculus efforts also funded by the National Science Foundation (e.g., at the elementary level see the *Patterns and Change* units of the TERC *Investigations* curriculum authored by Tierney, Nemirovsky, and Noble, 1995). This commitment to starting with the integral is one of four "major reversals" that has come to characterize the work of the SimCalc project on the mathematics of change (Kaput, 1994).

The following is a summarized version of the sequence successfully used in a number of adjusted forms with students in grades three through high school. The approach makes use of a simulation environment called *MathWorlds* (Kaput & Roschelle, 1996, 2000) that is available as a free download in both a Java™ version that will run on both Macintosh™ and Wintel machines and an earlier Mac-only version. In addition, there is a free shareware version available for the TI-83+ calculator. A more elaborate calculator version that includes some file management capabilities is also available for a small per-seat cost from SimCalc Technologies. The screen shots included in this article are from both the free Java-based computer version and the calculator version with the file management capabilities[1].

Starting with Physical Blocks

The delta block approach differs from real-time graphing environments, such as the widely used Calculator-Based Ranger™ (CBR) or Micro-Based Laboratory (MBL), in that the learners' attention is focused principally on the area under a velocity (rate) graph and not on the graph ("the line") itself. Using the *MathWorlds* software, the blocks of area can be arranged in various ways. Then when the simulation is run, the learner can explore the consequences these arrangements have for the motion of one or more simulated characters (e.g., the elevator in Figure 1).

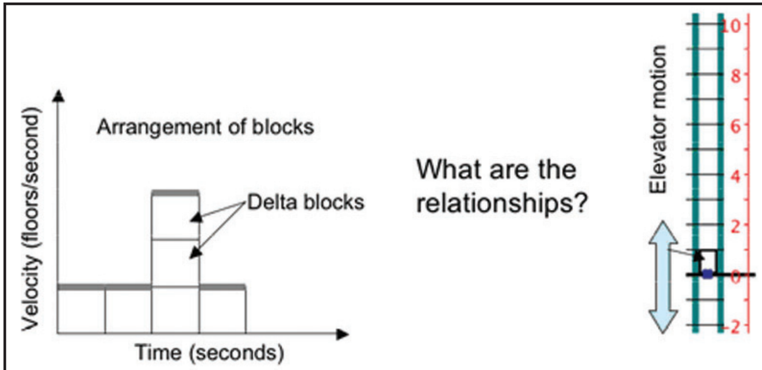

Figure 1. Arrangements of Blocks of Area Control the Motion of a Simulated Elevator

Using this approach the “line” or graph itself is thought of as just the top-most “edge” of this wall (see shaded top edge of the delta-block wall in Figure 1).

To support engagement with this approach, each student is given six physical blocks. The learners are then asked to arrange the blocks vertically in various ways according to some simple rules (see Figure 2).

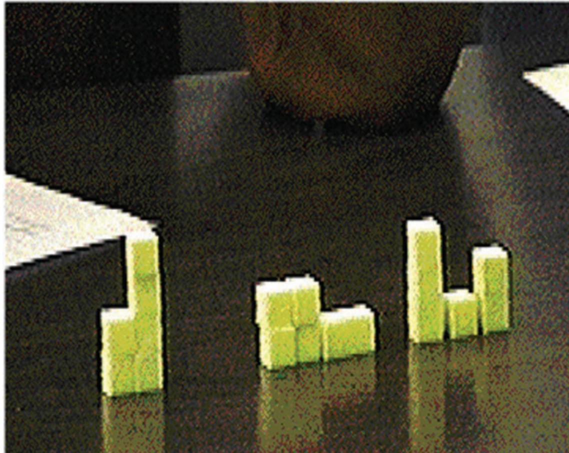

Figure 2. Student Arrangements of Six Physical Blocks

The rules are that the blocks have to be stacked straight on top of each other (no "straddlers"), with no vertical gaps (no "holes"), and no blocks hidden behind the wall (all the blocks must be visible from the front side).

A side-view of each student-generated wall is then shaded-in on graph paper. This shaded graph is used to support the transition to either the computer-based or the calculator-based *MathWorlds* simulation environment (Figure 3).

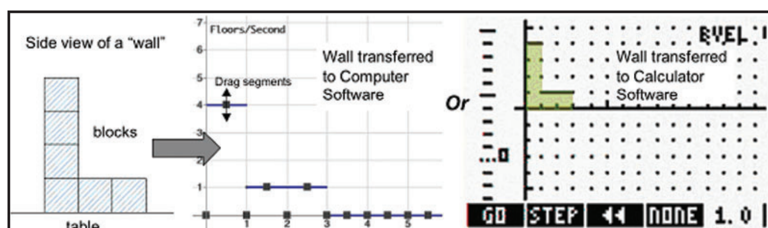

Figure 3. The Side-view of the Wall is Transferred to Either Computer or Calculator Software

The students transfer their walls into the computer software by "dragging" the given horizontal velocity segments up or down until the "wall" in the computer matches the shaded-in wall on their graph paper. Once there, students can run the simulation to explore how the motion of the elevator is related to different possible arrangements of the wall.

Note that in all of these explorations there are still six "blocks" of area under the graph. The graphical walls are linked to the arrangements of the original six physical blocks. Using this approach, students use examples they create to explore rate in a qualitative or comparative way. The approach is qualitative because rather than starting with ratio-based concepts and forms of measurement (e.g., "floors per second"), the students will start off talking in terms of relative speeds and in terms of "going" some number of floors "in" a given time interval. For the example shown in Figure 3, students notice that it goes "fast" for four floors (four floors *in* the first second) and "slow" for two floors. The core insight is that "it goes faster" because the wall is "higher" and then slower because the wall "is lower." Examples with varying rate help the students to understand the interpretation of the velocity graph. They also notice that the elevator, no matter what the shape of the block wall, always "goes six floors" from where it starts.

Challenges and Discovering Negative Blocks

After the students have entered their shapes, predicted what the elevator will do, and then checked to see if these descriptions are accurate, they are then given a series of challenges each of which has many possible solutions. The multiple solutions give the teacher many examples to use in the class discussions; they also provide some real-time feedback on how each student is doing. One of these early challenges asks the students to find, and then record on paper, two ways of having the elevator go to the third floor, stop to let some people off, and then go to the sixth floor. Interestingly, almost every time we have presented students with this challenge, at least a few students inadvertently discover that blocks "below" the axis make the elevator "go down." And, if not by accident, then some of the subsequent challenges have the students find solutions requiring the elevator to move downward. Figure 4 shows one possible solution to a challenge of finding multiple ways to have the elevator go to the third floor and then come back down to the starting floor.

Figure 4. Students Explore Positive and Negative Blocks as Additive Inverses

Students observe that the blocks below the axis make the elevator "go down" and that blocks below "cancel" with blocks above. For this example in Figure 4, the blocks below completely cancel with the blocks above (which can subsequently be represented, for an elevator starting on floor 0, as $0 + {}^{+}3 + {}^{-}1 + {}^{-}1 + {}^{-}1 = 0$ floor).

The software also allows students to change the initial position of the elevator. Similar to how they drag the graph segments, the students can select the elevator and drag it up or down. Then they run the simulation for the new initial position and explore how the motion is changed. If, for example, the elevator begins on $^{-}2$ floor (two floors "in the basement" as the students describe it), the graph in Figure 4 would have the elevator go up three floors (to the first floor) and then down three floors to return to its initial position (now summarized by the expression: $^{-}2 + {}^{+}3 + {}^{-}1 + {}^{-}1 + {}^{-}1 = {}^{-}2$ floor). The students come to see that these area blocks are really blocks of *change* – *delta* blocks – because they say something about how much the position of the elevator changes and not necessarily the floor on which the elevator ends up. Formally, this idea is connected with the constant of integration in calculus.

Quantifying "Fast-ness"

After developing these insights in the context of varying rate, the transition back to constant rate to develop ratio-based concepts and metrics (e.g., change in floors per second) can be explored. At this point, the students begin to feel a need to come up with a way of quantifying their comparative sense of "fast-ness." They move toward creating a number to represent this fastness as part of working with "walls" having uniform height (constant rate) and, in addition, by working with non-integer units of time. The area model of multiplication (side x side = area) and division (area ÷ either side = the other side) can be used to "solve" for the height of the uniform wall (the velocity). An elevator traveling six floors (area) in two seconds (horizontal side) has a height of three units (where, for these units, the language of three floors per second can be introduced). Similarly an elevator traveling three floors (area) in 1/2 second (horizontal side) has to have a height of six floors/second (students can adjust the height to get six as a complement to "doing the division" of 3 ÷ 1/2). A strength of the delta-blocks approach is that students start off reasoning in terms of blocks of area. In this way, fractional multiplication and division can be explored using the *MathWorlds* software. From there we can move to creating metrics for rate. That is, using the area model we can divide the area by the time to "get the number" or create a metric for the height of the rectangle (floors ÷ seconds or floors per second).

Position graphs are also available in the software. Therefore, the slope or "steep-ness" of a position-time graph can come to be associated with solving for the height of the velocity-time wall. Starting with qualitative rea-

soning about varying rate helps to support students' subsequent creation of metrics for rate and slope. This is because learners will have come to understand the distinct roles of "m," as related to the fast-ness of the elevator, and of "b," as the starting position of the elevator, in a way that supports their subsequent engagement with the formalisms of linear functions in slope-intercept form ($y = mx + b$) (Stroup 1996; Stroup, 2002).

Linear Velocity and Area

One of the critical questions at this point is whether the concepts of relative speed, area, and sign (blocks above canceling blocks below) can transfer to other more challenging contexts. Each velocity segment in the previous section was a piece-wise constant segment; the elevator would move at a constant velocity throughout each interval of time. Can students extend their insights developed with the delta-blocks to linear velocity segments (e.g., where velocity changes at a constant rate)?

Early on in our development of the delta-blocks approach, a sixth-grade student who had only completed the first portion of the sequence discussed earlier (up to the point of discovering that negative blocks made the elevator go down), had some free time in his day and came back to the math classroom. He wanted to continue to work with the software and announced to the teacher, as something of a challenge, that he understood this mathematics and that, "You can't stop me." What developed was that he and the teacher began giving each other challenges using the computer version of the software (he had worked with the calculator version in class). The teacher, who had taught in this inner city school for many years, later reported that his previous teachers had described him as a "D^{+}" math student. We had a camera available from videotaping the earlier class and we decided to turn it on for this exchange.

After a series of rounds in which the student and the teacher challenged each other, the following sequence of challenges with linear velocity segments was presented to the student. Although the teacher was well versed in using inquiry-based approaches to teaching, he had been a music major in college and was only just beginning his own formal study of calculus at a local university. Figure 5 depicts the sequence of three graphs with some of the student comments included.

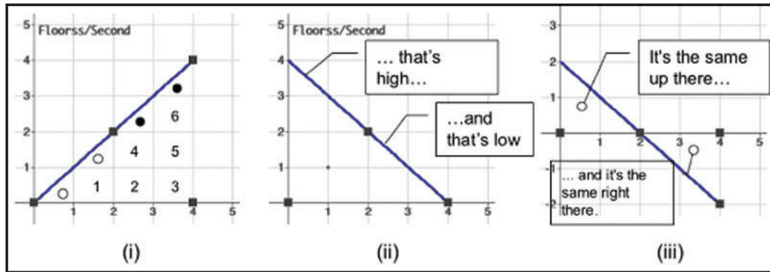

Figure 5. Sequence of Three Successive Velocity, Area, and Sign-related Episodes With a Sixth-Grade Learner who had Completed Some of the Delta-Blocks Challenges

Starting with the first graph (i) in Figure 5, the following is a transcription of the exchange.

Episode One

S: It's gonna go eight floors.
T: Eight floors ... why?
S: There's six floors right there [see 1-6 in (i) of Figure 5]. That's a floor [points with his index and middle finger to the two half-blocks shown with the open circles in (i) of Figure 5]. And, that's another floor [pointing to the remaining half-blocks with the solid circles in (i) of Figure 5].
T: So it's gonna go eight floors ... try it and see.

[The elevator moves slowly at first, speeds up, and then finishes on the eighth floor].

T: Ooh!, eight. And, what else did you notice about the elevator?
S: First it goes fast and then faster ... the last part is goin' real fast.
T: Okay, any idea why? What makes it do that?
S: No ...

Episode Two

T: What if I did this? [see (ii) in Figure 5]
S: Still eight!
T: Still eight?
S: Same problem.
T: Same problem, hungh? ... The elevator's gonna go the same way?

S: Yeah ... No! ... First it's gonna go fast and then slow.
T: Why?
S: 'Cause it's a different way. And, that's high and that's low. [He points to the respective regions of the graph shown in (ii) of Figure 5].
T: Try it, let's see.

[The elevator starts off fast and slows down to arrive at the eighth floor].

Episode Three

[Then, accidentally, the third graph (iii) in Figure 5 is created and the teacher asks the student to make sense of this graph].

S: Hey ... That's two, and there's some down on the bottom. Can't do that. [The student starts to move the linear velocity segment upward].
T: I donno, try it. Bring it back down.
S: Wha ... [The student moves the graph back down to (iii) in Figure 5]?
T: What do you think's gonna happen there?
S: [Pause]. It's gonna go back to zero after it's finished.
T: You think so.
S: Yeah.
T: Try it, let's see ... what, what, wait before you do that. Why do you think it's gonna go back to zero?
S: [Touching the screen twice as shown in (iii) of Figure 5] It's the same up there and it's the same right there. They're both the same.
T: Okay, try it. I donno. [This teacher's pedagogical style is to say "I donno" to encourage the student's own sense making].

[The elevator goes up two floors and then comes back to zero].

T: Good, good...

The last graph, in particular, is the kind of representation of constant acceleration a student would typically expect to study in the kinematics portion of a high school physics course. That said, a number of kinds of basic mathematics are also used by this student. He clearly reasons in terms of the blocks and extends this reasoning to linear velocity graphs and fractional areas (e.g., combining the half-blocks in the first episode). In addition, even though the graph is no longer piece-wise constant, the height of the graph still is associated with the relative speed of the elevator. He notes for (ii)

in Figure 5 that "First it's gonna go fast and then slow" and associates this with "that's high and that's low" on the graph. He accurately solves for the area of the triangles. And, for the third graph he integrates his understanding of area with sign and notes that the triangular areas above and below cancel ("there's two" and "[t]hey're both the same" so it's "gonna go back to zero"]. He was able to extend insights from the delta-blocks introduction and he did so in ways that simultaneously used basics and calculus. The calculus context served to situate and make his use of the basics meaningful in a way that he found both accessible and powerful.

Using More than One Elevator to Compare Motions

The *MathWorlds* software also readily supports the simultaneous motion of more than one simulated character (e.g., elevators but also cars and other characters). By using this feature, and working with both piece-wise constant and linear velocity segments, a number of standard geometric figures can be drawn. Calculating the area under the curves allows the students to predict how far (or far apart) the respective elevators will move. The power of this can be illustrated for the "classic" challenge of having students truly believe that triangles having the same base and height will have the same area. Figure 6 shows two elevators and their respective graphs. In this case, the triangles (see (i) in Figure 6) are set up so that they have the same base (four seconds) and height (five floors/second). We then ask the students: "If these elevators were racing, which one would finish first or would they finish at the same time?" Usually the students divide into which-is-first camps instead of believing they would finish at the same time (i.e., the same bases and the same heights should mean that while they may move in distinct ways, they will still cover the same displacement in the same time). When pushed, an animated discussion ensues with most students believing that the graph that gets highest (fastest) first will be ahead and thus finish first (this is the left elevator in Figure 6).

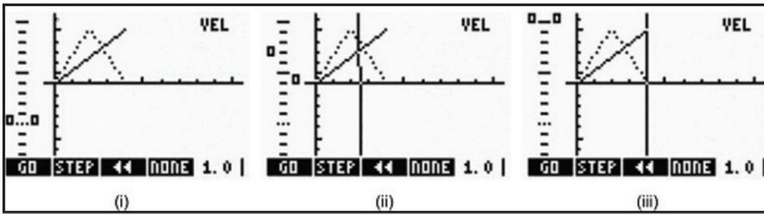

Figure 6. Students are Surprised When These Two Graphical Triangles Really do Travel the Same Distance in the Same Amount of Time

When the simulation is now run, the students in the fastest-first camp start off believing they are right because this elevator does in fact take "an early lead" (see (ii) in Figure 6). However, by the end, (see (iii)) things get more complicated. Often the simulation needs to be run multiple times and/or a "step mode" feature in the software must be used to jump through the motion in one-second intervals. The students do not really doubt that the formula (1/2 * base * height) gives the same "answer." Instead, this is one of those times that simply showing this formulaic result does not really convince them[2]. They need to find a way, relative to the graphs and the motions of the elevators, to see that this formula makes sense in the context of varying rates. In addition, as was true for the example with the sixth-grader, they will need to draw on all sorts of "basic" knowledge—including dividing up the areas and reassembling them—to convince themselves that what they "see" really does make sense.

"What's the Difference?" and Subtraction

Comparisons of the final positions of simulated characters can be used to explore "difference" as a way of understanding subtraction. There are many ways in which subtraction can be discussed in the standard curriculum. Perhaps the most common in elementary and middle school contexts is "take away." To help with sign (+/-), bi-colored tiles often are used to represent positive and negative numbers, and the operation of addition (as putting in), and the operation of subtraction (as taking away). Critical to the use of these tiles is for students to understand that a positive tile will cancel with a negative tile (the tiles are additive inverses). The earlier use of positive and negative delta blocks (blocks above and below the axis respectively) has helped teachers find a way of illustrating this idea of additive inverse (moving up one floor and then down one "cancels" or is the same as doing

nothing—see the circles in Figure 4 above). As the teachers we have worked with have noted, this insight alone is often enough to support students' subsequent use of the bi-colored tiles to further explore the various combinations of adding and subtracting integers.

However, the use of delta-blocks can also help students explore another way of thinking of subtraction. As an alternative to thinking of subtraction as taking away, subtraction also can be thought of as a kind of comparison. In everyday speech we often ask, "What's the difference?" between two quantities. Examples include comparisons of the heights of two students or, in another context, what sometimes is called the "spread" on a basketball game. In these contexts, subtraction models what we mean by "what's the difference?" between two quantities, A and B.

Figure 7 shows a series of "What's the difference?" situations. If we ask what is the difference between 5 and 3, students will say 2. Five-foot tall Sally can be said to be two feet taller than three-foot Sandra. If we represent this with mathematical notation, A – B, then 5 feet – 3 feet = 2 feet. Allowing for sign, if we ask for a comparison of Sandra's height with Sally's, we could say Sandra is two feet shorter, or 3 feet – 5 feet = ($^{-}2$) feet. Using delta-blocks these first two comparisons can be related to comparisons of the motions (from the same starting position) of two elevators (see (i) and (ii) in Figure 7).

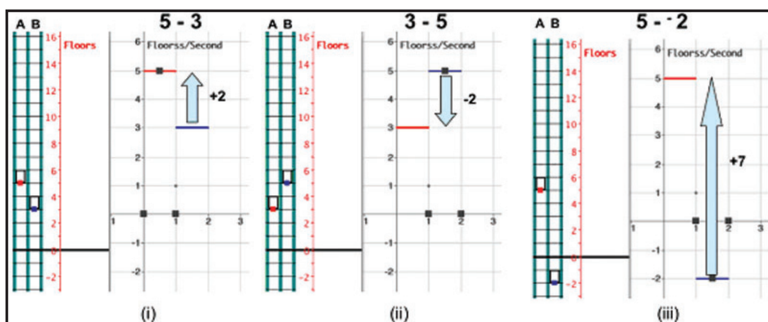

Figure 7. Subtraction Treated as a Comparison of A and B

Graphs (i) and (ii) show how subtraction can represent the pattern of asking how far the second quantity is from the first quantity, $^{+}2$ floors in the first case and $^{-}2$ floors in the second. The "Holy Grail" of teaching signed subtraction is "subtracting a negative." In this context, we can ask what is the difference between positive 5 and negative 2. Graph (iii) in Figure 7 shows the result for (5 – ($^{-}2$)): The second quantity is $^{+}7$ from the first.

Therefore, $(5 - (^{-}2))$ is $^{+}7$. Again, subtraction, as a comparison of two quantities, can be embodied in this calculus-related way. This approach can be added to a teacher's repertoire for exploring the basics of subtraction in a way that invites the learner to move beyond simply memorizing rules such as "subtracting a negative is the same as adding a positive."

Area and Invariance

An interesting extension of some of these ideas of difference, geometric area, and piece-wise constant, as well as linear velocity, is to ask students to say what it would mean for the motion of the elevators if the graphs of two elevators form a geometric figure such as a trapezoid as shown in (i) in Figure 8. We then explore what would happen if this same geometric figure is moved up or down as is shown in (iii) of Figure 8.

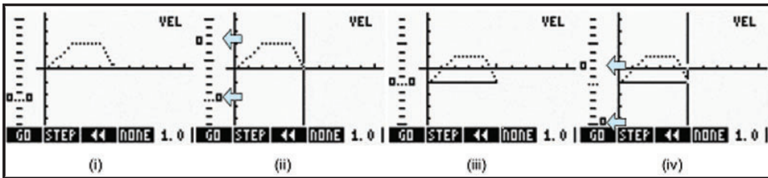

Figure 8. An Exploration of What Stays the Same About the Motion of Two Elevators When the Same Geometric Area is Moved

In each case, (i) or (iii), the "base" of the trapezoid is the velocity graph of the second elevator. The remaining lines of the trapezoid comprise the velocity graph for the first elevator. When these two graphs of motion are combined on the same axis, they can be seen to form a closed trapezoidal figure. Given that these trapezoids are constructed to be the same shape, the question then becomes what remains the same about the motions of the respective pairs of elevators.

Screen shots (i) and (ii) show the positions of the elevators before and after the simulation is run for the first trapezoid (the base graph for the second elevator is not visible because it lies along the horizontal axis). For this first trapezoid, the elevators for both start on floor 0 (see i) and end up on floors 7 and 0 respectively (see ii). During the simulated motion, the first elevator moves upward at differing speeds during the distinct regions of the velocity-time graph that controls its motion. The second elevator does not move because it has a constant, zero-velocity graph. The elevators end up seven floors apart.

When the students examine the graphs carefully, they readily see that

the graphs forming the trapezoids in (i) and (iii) of Figure 8 have the same basic shape and area. It is interesting, however, that when the simulations are run for these pairs of elevators, the respective motions appear very different. For (i), the first elevator moves in a positive direction for the entire time and the second elevator does not move at all, remaining at floor 0. The resulting positions for this pair of elevators are shown in (ii). For (iii), the elevator on the left begins by moving downward, then moves upward, and then down again toward the end of the motion. At the same time, the elevator on the right in (iii) does not stay stationary but instead moves downward at a constant negative velocity. Clearly, these pairs of motions are different in terms of how the respective elevators move. The question then becomes what remains the same about motions? More formally, what form of invariance corresponds to the area of the trapezoid remaining the same? After experimenting with running the respective simulations several times, one student noticed, "The difference is the same." She went on to explain that the corresponding differences in the final positions of the elevators remain "the same" — seven floors. In (ii), the finishing positions are floor 7 and floor 0. In (iv) the corresponding finishing positions are $^{+}2$ and $^{-}5$. Both final positions have a "difference" in final position of seven floors.

At this point, powerful ideas related to the formal study of calculus (i.e., the area or integral "between" curves) become more thoroughly interwoven with some of the "basics" concepts students can build on from using the approaches discussed earlier. The mathematically significant idea of area is, in some sense, freed from always having to be about discussing surfaces such as tabletops or sides of buildings. Mathematics can start to be viewed as a systematic study of these types of "same-ness" or invariance. Mathematical invariance is what allows us to see commonality in many situations that, on the surface, may appear to be very different, e.g., the very distinct up-and-down motions of the pairs of elevators depicted in Figure 8. This insight is, in itself, a significant "meta-lesson" that can be drawn from seeing "basics" concepts such as area as deeply interwoven with what are often considered "advanced" topics such as calculus.

CONCLUSION

This article set out to suggest that using mathematical contexts closely associated with the study of calculus can be helpful, in very practical ways, in supporting our students' developing understandings of what we often refer to as "basics" in our current curricula. In other work we discuss the relationships between young students' emerging insights into the relationships between rate (how fast) and amount (how much), and the formal study of

calculus (Stroup, 2002). We can also note in passing that many other similar, calculus-related, examples can be provided that are related to topics such as data analysis (e.g., histograms), probability, and other core topics in the curriculum. However, the purpose of this article is to serve as an introduction to the possibility of calling into question some of our fundamental assumptions about how the curriculum is organized. The idea is to provide an overview of some approaches that literally could be used "tomorrow" in math classes using readily available, and in some cases free, software running on machines that many elementary and middle schools already have available. There need not be a tension between learning the basics and using simulation environments to introduce advanced mathematics early in the curriculum.

Experience has shown that the leveraging of calculus in this way does in fact help with teaching and learning the basics. More systematic study on a large scale is now being planned. However, the students in a very high-poverty school in Roxbury, Massachusetts, who as sixth-graders used a curriculum that included these activities and emphasized a function-based approach to algebra, ended up scoring, as a class, near the top in district-wide, high-stakes assessments. In Texas, similar results with these materials have been obtained for groups of students who were predicted to fail the final exam in algebra and ended up, after a one-semester intervention, obtaining a passing rate that matched that of students in the rest of the state (Brawner, 2001).

Moving beyond the practical use of the these activities in classes, however, this article also has the not-so-hidden agenda of supporting one of the critical features of the reform movement — namely, that "big ideas" in mathematics curricula can and should be viewed as strands to be woven together; not as a layer-cake of separate topics to be pushed through in getting to the "really interesting stuff" such as calculus.

What is outlined above need not be viewed as a replacement for a standard treatment of calculus. At the very least, however, these approaches will certainly leave more students in a better position to understand what the formalisms of calculus are all about, and thereby, potentially increase their interest in continuing their study of mathematics through high school and beyond. As important, however, may be the goals of advancing a deeper sense of how areas of mathematics work together while also opening up the possibility that some fundamental ideas of calculus can be part of all our students' mathematical repertoire (cf., Kaput, 1994). Rather than limiting access to the mathematics of change to the still-too-few students who study calculus toward the end of the secondary curriculum, we can now view "teaching the

basics" as an occasion to simultaneously introduce "big ideas" related to the mathematics of change. It is not "basics" versus "advanced" topics such as calculus (or algebra, for that matter). Basics and advanced topics can work together in providing all our students with meaningful and powerful ways of understanding the world, including the world of mathematics proper.

Acknowledgements

Critical to the development of this line of research is previous and ongoing funding from the National Science Foundation including the following grants: The Northeastern University Comprehensive Regional Center for Minorities (David Blackman, PI), The SimCalc Project (James J. Kaput and Jeremy Roschelle, PIs), CAREER: Learning Entropy and Energy Project (Walter M. Stroup, PI). The Boston affiliate of Bob Moses' Algebra Project supported significant aspects of the site-based work and teacher professional development (Jacqueline Rivers, Executive Director). Texas Instruments also has generously supported this work. We gratefully acknowledge this support. The views expressed herein are those of the author and do not necessarily reflect those of the funding institutions. I also want to acknowledge James A. Early, as a master teacher and valued colleague, in this work.

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Notes

[1] The specific files and links to relevant sites can be found at http://128.83.243.140/calculus.htm

[2] In passing, it is worth noting that in professional development settings a number of very experienced teachers also have had an "ah-ha" moment of recognition as well. This is a new way of seeing an old "basics" problem.